Gamma-Ray Production From Cosmic Electron-Positron Annihilation

2-1 CROSS SECTIONS AND ANNIHILATION RATES

As is well known, electrons and positrons may interact electromagnetically and thereby annihilate to produce γ-rays. This annihilation may occur in the following ways.[1]

(1) A free positron may annihilate with a free electron to produce, most frequently, two γ-rays; i.e.,

$$e^+ + e^- \rightarrow \gamma + \gamma \tag{2-1}$$

(2) A positron and electron of low (typically thermal) energy may first combine to form a hydrogenlike system known as positronium. The positronium may then annihilate, usually into two or three γ-rays.

We can discuss these annihilations with much more confidence and exactness than we can discuss the γ-ray production from high-energy inelastic collisions (as we will in a later chapter) because here we can use the results of quantum electrodynamics, which we know to be a very successful theory, whereas there is presently no really good theoretical understanding of particle production in high-energy interactions. Consequently, we will be forced to use a phenomenological description in the discussion of the high-energy processes.

2-2 THE FREE-ANNIHILATION GAMMA-RAY SPECTRUM

The cross section for free positron annihilation as a function of energy was first deduced by Dirac (1930). An excellent presentation of the theory

[1] Positrons may annihilate with bound electrons to produce a single photon, with momentum being conserved by the binding nucleus. However, the cross section for this process is always less than $\pi r_0^2 Z^5/(137)^4$, where the atomic number Z is almost always 1 or 2 under astrophysical conditions. Therefore, the single-photon annihilation can be neglected in the following discussion (Heitler, 1960).

is given by Heitler (1960). The most important annihilation mode of the free electron-positron system is the annihilation

$$e^+ + e^- \rightarrow \gamma + \gamma \tag{2-1}$$

The frequency of this annihilation is 372 times greater than that of the free three-photon annihilation and we may neglect all but the two-photon mode in our consideration of the γ-ray spectrum from free $e^+ - e^-$ annihilations. (However, as we shall see later, the three-photon mode becomes important when we consider the effect of positronium formation by positrons of energies less than 5 keV.)

The differential cross section for γ-ray production in the collision cms of a free two-photon annihilation may be written as

$$d\sigma = \frac{\sigma_0}{2\gamma_c^2 \beta_c} \phi(\chi|\gamma) d\chi \tag{2-2}$$

where $\sigma_0 = \pi r_0^2 = 2.5 \times 10^{-25}$ cm^2, using the definition of the classical radius of the electron, $r_0 = \frac{e^2}{m_e c^2} = 2.8 \times 10^{-13}$ cm; γ is the Lorentz factor of the positron; $\beta = \left[1 - \left(\frac{1}{\gamma^2}\right)\right]^{1/2}$; the cms Lorentz factor and the velocity are given by

$$\gamma_c = \left(\frac{\gamma + 1}{2}\right) \qquad \text{and} \qquad \beta_c = \left(\frac{\gamma - 1}{\gamma + 1}\right)^{1/2} \tag{2-3}$$

χ is the cosine of the angle between the incoming positron and the outgoing γ-ray in the cms; and $\phi(\chi|\gamma)$, the angular distribution function, is defined as

$$\phi(\chi|\gamma) = \frac{1 + \beta_c^2(2 - \chi^2)}{1 - \beta_c^2 \chi^2} - \frac{2\beta_c^4 (1 - \chi^2)^2}{(1 - \beta_c^2 \chi^2)^2} \tag{2-4}$$

The energy of an annihilation γ-ray in the ls is given by the Doppler relation as

$$E_\gamma = (m_e c^2)\ \gamma_c^2 (1 + \beta_c \chi) \tag{2-5}$$

If we now define the dimensionless energy $\eta = E_\gamma / m_e c^2$, we may use equations (2–3) and (2–5) to determine the normalized distribution func-

tion that uniquely relates χ to the laboratory Lorentz factor of the positron and the laboratory energy of the γ-ray as follows:

$$f(\chi|\gamma, \eta) = 2(\gamma^2 - 1)^{-1/2}\delta[\chi - \chi_0(\eta, \gamma)]$$

where

$$\chi_0(\eta, \gamma) \equiv \frac{(2\eta - 1) - \gamma}{(\gamma^2 - 1)^{1/2}} \tag{2-6}$$

The total number of γ-rays produced per second by a positron of energy $\gamma m_e c^2$ is

$$Q_{\text{total}}(\gamma) = \frac{2\sigma_0 n_e \beta c}{2\gamma_c^2 \beta_c} \int_{-1}^{1} d\chi\, \phi(\chi|\gamma)$$

$$= \frac{2\sigma_0 n_e c}{\gamma} \int_{-1}^{1} d\chi\, \phi(\chi|\gamma)$$

where n_e is the number density of electrons in the medium.

Therefore, the γ-ray source spectrum from the annihilation of cosmic-ray positrons with a density $n(\gamma)$ cm^{-3} is given by

$$Q(\eta) = 4n_e\sigma_0 c \int_1^{\infty} d\gamma \frac{n(\gamma)}{\gamma(\gamma^2 - 1)^{1/2}} \int_{-1}^{1} d\chi\, \phi(\chi|\gamma)\delta[\chi - \chi_0(\eta, \gamma)] \tag{2-7}$$

We now wish to integrate over the delta function in equation (2–7). The result of this integration is to replace the last integral in equation (2–7) by a function $\Phi(\eta, \gamma)$ such that

$$\Phi(\eta, \gamma) = \begin{cases} \phi[\chi_0(\eta, \gamma), \gamma] & \text{for} \quad |\chi_0| \leq 1 \\ 0 & \text{for} \quad |\chi_0| > 1 \end{cases} \tag{2-8}$$

We may, therefore, replace the integral over $d\chi$ by the algebraic function $\phi[\chi_0(\eta, \gamma), \gamma]$, provided the limiting condition on χ_0 is transformed into a limiting condition on the integration over $d\gamma$. The limiting conditions $\chi_0 = \pm 1$ correspond to the relations

$$\eta_{\pm} = \gamma_c^2(1 \pm \beta_c) \tag{2-9}$$

It follows from equation (2–9) that the product

$$\eta_+\eta_- = \gamma_c^4(1 - \beta_c^2) = \gamma_c^2 = \tfrac{1}{2}(\gamma + 1) \tag{2-10}$$

and the sum

$$\eta_+ + \eta_- = 2\gamma_c^2 = \gamma + 1 \tag{2-11}$$

Therefore, η_+ and η_- are the roots of the quadratic equation

$$\eta^2 - 2\gamma_c^2\eta + \gamma_c^2 = 0 \tag{2-12}$$

which can be solved for γ_c in terms of η to give

$$\gamma_c^2 = \frac{\eta^2}{2\eta - 1} = \tfrac{1}{2}(\gamma + 1) \tag{2-13}$$

or

$$\gamma = \frac{\eta^2 + (\eta - 1)^2}{2\eta - 1} \tag{2-14}$$

which is finite and positive for $\eta > 1/2$ and has a minimum of $\gamma = 1$ at $\eta = 1$.

Figure 2–1 shows the curve defined by equation (2–14) split into the branches defined by the physical extremes $\chi = \pm 1$. Also shown are the asymptotes $\eta = 1/2$ and $\eta = \gamma + 1/2$, the line defining $\chi = 0$, and the shaded region corresponding to the physical values of $|\chi| \leq 1$. It can be

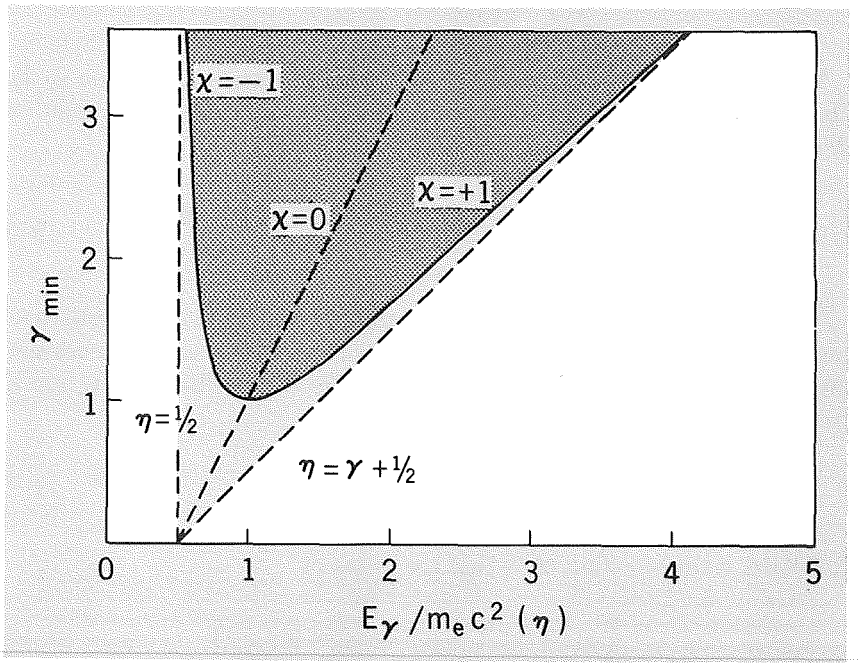

FIGURE 2–1.—The kinematic limits on the positron Lorentz factor involved in the determination of the laboratory energies of the annihilation γ-rays produced (Stecker, 1969a).

seen from figure 2–1 that the physical range of γ defined by the shaded region ($|\chi| \leq 1$) is bounded on the bottom by the curve of equation (2–14) and is unbounded on the top.

Figure 2–1 indicates that no γ-ray can be produced from an annihilation process that has an energy less than or equal to $\frac{1}{2}m_e c^2$. This physical restriction may be seen more clearly as a direct consequence of equation (2–9), since

$$\eta_- \leq \eta \leq \eta_+ \tag{2–15}$$

where

$$\eta_- = \gamma_c^2(1-\beta_c) = \frac{1-\beta_c}{1-\beta_c^2} = \frac{1}{1+\beta_c} > \frac{1}{2} \tag{2–16}$$

Thus η_- may approach, but never reach 1/2 as $\gamma_c \to \infty$. On the other hand

$$\eta_+ = \gamma_c^2(1+\beta_c) = \frac{1}{1-\beta_c} \tag{2–17}$$

which increases without bound as $\gamma_c \to \infty$.

The general restriction on the range of η may be designated by the introduction of the Heaviside step function $H_+(\eta_0)$, which is defined by the relation

$$H_+(\eta_0) = \begin{cases} 1 & \text{for } \eta > \eta_0 \\ 0 & \text{for } \eta \leq \eta_0 \end{cases} \tag{2–18}$$

We may therefore rewrite equation (2–7) in the form

$$Q(\eta) = 4H_+\left(\frac{1}{2}\right) n_{e^-}\sigma_0 c \int_{G(\eta)}^{\infty} d\gamma \, \frac{n(\gamma)}{\gamma(\gamma^2-1)^{1/2}} \, \phi[\chi_0(\eta, \gamma), \gamma] \tag{2–19}$$

where

$$G(\eta) = \frac{\eta^2 + (\eta-1)^2}{2\eta - 1} \tag{2–20}$$

From the general results which we have obtained, we can immediately arrive at formulas for asymptotic spectra that can be used as guides in the examination of the results of numerical calculations. These are obtained as follows:

(1) For two-photon annihilations at rest, it follows from equation (2–5) that the annihilation γ-ray spectrum (AGS) is simply a line at energy $\eta = 1$ (0.51 MeV). This is, of course, a familiar and expected result.

(2) The AGS from the two-photon annihilation of an ultrarelativistic positron may be obtained from a consideration of the angular

distribution function $\phi(\chi|\gamma)$, given by equation (2–4). At ultrarelativistic energies, the angular distribution function peaks sharply at $\chi = \pm 1$ so that the γ-rays resulting from the annihilation lie close to the asymptotes of figure 2–1; viz, $\eta = 1/2$ and $\eta = \gamma + 1/2$. This result, as pointed out by Heitler (1960), may be discussed physically as follows: In an ultrarelativistic annihilation, the resultant photons are emitted in a sharply backward and sharply forward direction in the collision cms. In the ls, the forward photon carries off almost all the energy of the collision and the backward photon carries off an energy between about 0.25 and 0.5 MeV. Therefore, the AGS for $\eta \gg 1$ may be obtained by using the approximate production cross section

$$\sigma_A(\eta|\gamma) \simeq \sigma_A(\gamma)\delta(\eta - \gamma) \qquad \text{for} \qquad \eta \gg 1 \tag{2–21}$$

The total cross section for free annihilation of positrons into γ-ray pairs was first calculated by Dirac (1930) and is given by

$$\sigma_A = \frac{\sigma_0}{\gamma+1}\left[\frac{\gamma^2+4\gamma+1}{(\gamma^2-1)^{1/2}} \ln\left[\gamma+(\gamma^2-1)^{1/2}\right] - \frac{\gamma+3}{(\gamma^2-1)^{1/2}}\right] \tag{2–22}$$

At nonrelativistic energies ($\gamma \simeq 1$), equation (2–22) reduces to the asymptotic form

$$\sigma_A \rightarrow \frac{\sigma_0\gamma}{(\gamma^2-1)^{1/2}} \rightarrow \frac{\sigma_0}{\beta} \qquad (\beta \ll 1) \tag{2–23}$$

At extreme relativistic energies ($\gamma \gg 1$), equations (2–21) and (2–22) reduce to the asymptotic form

$$\sigma_A \rightarrow \sigma_0\left[\frac{\ln(2\gamma)-1}{\gamma}\right] \qquad (\gamma \gg 1) \tag{2–24}$$

and $Q(\eta) \rightarrow n_e\sigma_0 cn(\eta)[\ln(2\eta) - 1]/\eta$ as has been noted by Ginzburg and Syrovatskii (1964). A free positron and free electron may also annihilate to produce three or more γ-rays. The cross section for the process

$$e^+ + e^- \rightarrow \zeta\gamma \tag{2–25}$$

is of the order

$$\sigma_{A,\,\zeta\gamma} \simeq \alpha^{(\zeta-2)}\sigma_{A,\,2\gamma} \tag{2–26}$$

where

$$\alpha = \frac{e^2}{\hbar c} \simeq \frac{1}{137} \tag{2–27}$$

The cross section for three-photon annihilation in the nonrelativistic limit ($\beta \ll 1$) is also inversely proportional to β and is given by Ore and Powell (1949).

$$\sigma_{A,3\gamma} \rightarrow \frac{4}{3}(\pi^2 - 9)\alpha \frac{r_0^2}{\beta} \qquad \beta \ll 1$$
$$\simeq \frac{\sigma_{A,2\gamma}}{372} \tag{2-28}$$

In our discussion we can neglect processes involving $\zeta > 3$, where cross sections contain increasingly large factors of α.

2–3 THE ANNIHILATION GAMMA-RAY SPECTRUM FROM POSITRONIUM FORMATION

A positron and electron of low (typically thermal) energy may first combine to form a hydrogenlike system known as positronium (which we will here designate by the symbol Π). The positronium may then annihilate into γ-rays. We must therefore consider processes of the type

$$e^+ + e^- \rightarrow \Pi \qquad \Pi \rightarrow \zeta\gamma \tag{2-29}$$

Here again, we can neglect processes involving $\zeta > 3$.

The positronium system, which is similar in structure to the hydrogen atom, can exist in the ground state ($L = 0$) either as a triplet ($^3S_1$), called orthopositronium by analogy with the spectral designation for hydrogen, or as a singlet ($^1S_0$) called parapositronium (also by analogy with hydrogen). In the triplet state, the spins of the electron and positron are parallel, whereas in the singlet state they are antiparallel. Since the positronium system has well-defined values of the quantum numbers L and S and has no net charge, the system is in an eigenstate of charge conjugation.

The positronium annihilations, unlike the free annihilations, obey selection rules since positronium is in an eigenstate of charge conjugation C, and C is conserved in electromagnetic interactions.

If ζ photons are produced in the final state, then

$$C = (-1)^\zeta \tag{2-30}$$

It can be shown that under particle exchange, positronium obeys a kind of generalized Pauli principle and changes sign. The exchange of

particles involves the exchange of the product of the space, spin, and charge-conjugation parts of the wave function. Therefore,

$$(-1)^{l}(-1)^{S+1}C=-1 \tag{2-31}$$

where l is the orbital angular-momentum quantum number and S is the spin quantum number. By combining equations (2–30) and (2–31), we obtain the selection rule

$$l+S=\zeta \tag{2-32}$$

We may therefore specify the processes given by equation (2–29) as

$$e^{+}+e^{-}\rightarrow \Pi(^{1}S_{0}) \hookrightarrow 2\gamma \tag{2-33}$$

and

$$e^{+}+e^{-}\rightarrow \Pi(^{3}S_{1}) \hookrightarrow 3\gamma \tag{2-34}$$

Processes involving $l\neq 0$ can be neglected (Deutsch, 1953).

The positronium annihilation rate, or inverse lifetime, is equal to the product of four times the free-annihilation cross section, the relative velocity of the electron and positron, and the overlap probability of the electron and positron wave functions. The factor of 4 is required because the free-annihilation cross section is an average over all four possible relative spin orientations of the positron and electron; viz, up-up, up-down, down-up, and down-down; whereas the positronium system annihilates from a definite orientation state.

If we designate the probability density of the wave function as the square of its absolute value $|\psi|^2$ in the usual notation, we then find

$$\begin{aligned}\tau_{2\gamma}^{-1}&=4\sigma_{A,2\gamma}v|\psi(0)|^{2}\\&=4\pi r_{0}^{2}\frac{c}{v}v\frac{1}{\pi}\left(\frac{1}{2a_{0}}\right)^{3}\\&=\frac{r_{0}^{2}c}{2a_{0}^{3}}=\alpha^{4}\frac{c}{2a_{0}}\end{aligned} \tag{2-35}$$

or

$$\tau_{2\gamma}=\alpha^{-4}\frac{2a_{0}}{c}=1.25\times 10^{-10}\ \text{s} \tag{2.36}$$

where a_0 is the Bohr radius given by

$$a_{0}=\alpha^{-2}r_{0}=5.29\times 10^{-13}\ \text{cm} \tag{2-37}$$

v is the electron velocity and $|\psi(0)|$ is the absolute magnitude of the wave function at the origin of the system. The effective Bohr radius for positronium is twice that of hydrogen, and this is taken into account. The factor $|\psi(0)|^2$ is just the effective electron density seen by the positron.

The lifetime for positronium annihilation into three γ-rays is derived from the free-annihilation cross section given by equation (2–28) through the use of the same method as that used in the derivation of equation (2–36). We thus obtain (Ore and Powell, 1949)

$$\tau^{-1}_{^3S_1 \to 3\gamma} = \frac{2}{9\pi}(\pi^2 - 9)\frac{\alpha^5 c}{a_0} \tag{2–38}$$

or

$$\tau_{^3S_1 \to 3\gamma} = 1.4 \times 10^{-7}\ \text{s} \tag{2–39}$$

This result can also be expressed as

$$\frac{\sigma_{f,3\gamma}}{\sigma_{f,2\gamma}} = 3\,\frac{\tau^{-1}_{^3S_1 \to 3\gamma}}{\tau^{-1}_{^1S_0 \to 2\gamma}} \tag{2–40}$$

Here again, the factor of 3 in equation (2–40) expresses the difference in probability between a singlet and triplet interaction (3/4 versus 1/4).

The lifetimes for both the two- and three-photon annihilations of positronium are so short that for galactic γ-ray production, we may consider the annihilation to take place effectively at the time that positronium is formed. Since the probability for positronium formation in the triplet state is three times the probability for positronium formation in the singlet state, it follows from equations (2–33) and (2–34) that 3/4 of the positronium formed in the galaxy undergoes three-photon annihilation.

2–4 THE GAMMA-RAY SPECTRA FROM ELECTRON-POSITRON ANNIHILATION

The energy spectrum of the two-photon annihilation in the cms of the electron-positron pair is a single line at $E_\gamma = m_e \simeq 0.511$ MeV, as can easily be seen from considerations of conservation of energy and momentum. The natural width of this line due to the uncertainty principle is small, being on the order of

$$\Delta E = \frac{\hbar}{\tau} \simeq 5.3 \times 10^{-12}\ \text{MeV} \tag{2–41}$$

Dominant broadening can be expected to be due to the Doppler effect, and is of the order

$$\frac{\Delta E}{E} = \frac{\Delta E}{m_e c^2} = \beta \tag{2-42}$$

The effect of astrophysical conditions on the broadening of the 0.51-MeV line from two-photon annihilation can be determined as follows:

In free e^+–e^- annihilations, we may consider a gas or plasma at temperature T. Then the distribution of the component of particle velocity along the line of sight of the observed γ-ray is of the form

$$f(\beta_{||})\,d\beta_{||} = \left(\frac{b}{\pi}\right)^{1/2} e^{-b\beta_{||}^2} d\beta_{||} \tag{2-43}$$

where

$$b \equiv \frac{m_e c^2}{2kT} \tag{2-44}$$

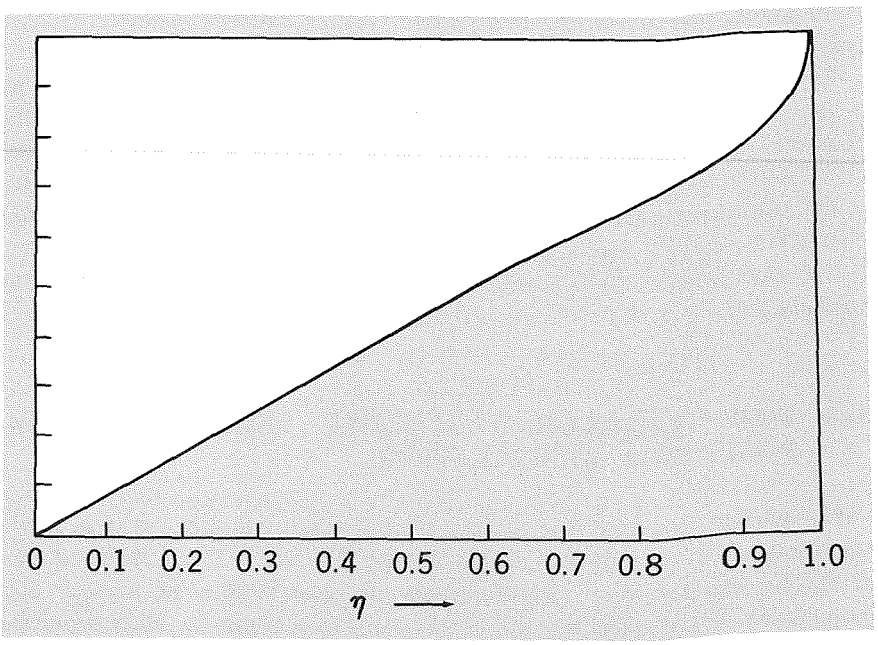

FIGURE 2–2.—Energy spectrum of γ-rays resulting from the three-photon annihilation of an electron and a positron. The abscissa is the photon energy in units of $m_e c^2$, and the ordinate is proportional to the number of photons per unit energy interval (from Ore and Powell, 1949).

It follows from equations (2–42) and (2–43) that the spectral-line shape has the gaussian form

$$f(\eta)\ d\eta = \left(\frac{b}{\pi}\right)^{1/2} \exp\left[-b\left(\frac{\eta-1}{\eta}\right)^2\right] d\eta \qquad (2\text{–}45)$$

Equation (2–45) has the same form as equation (2–43), since the number of collisions involving velocity v is proportional to $\sigma v \simeq \sigma_0 c$, independent of v.

The magnitude of the broadening is of the order

$$\Delta\eta = \left(\frac{\Delta E_\gamma}{m_e c^2}\right) \simeq b^{-1/2} \simeq 1.8 \times 10^{-5} T^{1/2} \qquad (2\text{–}46)$$

so that for $T = 100$ K, $\Delta E \simeq 10^{-1}$ keV. However, we have shown in the last section that most of the annihilations near rest occur following the formation of positronium by positrons having an average energy of about 35 eV. It therefore follows from the results of the previous section and equation (2–42) that 75 percent of the resultant annihilations occur via the three-photon channel and 25 percent produce a two-photon line with a Doppler width of about 5 keV.

The energy spectrum from the three-photon annihilation is continuous, as allowed by conservation of momentum. It has been calculated by Ore and Powell (1949) to be of the form

$$F(\eta) = \frac{2}{\pi^2 - 9}\left[\frac{\eta(1-\eta)}{(2-\eta)^2} - \frac{2(1-\eta)^2}{(2-\eta)^3}\ln(1-\eta) + \frac{2-\eta}{\eta} + \frac{2(1-\eta)}{\eta^2}\ln(1-\eta)\right] \qquad (2\text{–}47)$$

The function $F(\eta)$ is shown in figure (2–2). $F(\eta)$ is normalized so that

$$\int_0^1 F(\eta)\ d\eta = 1 \qquad (2\text{–}48)$$